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LETTER TO THE EDITOR

Precise determination of the fractal dimensions of Apollonian packing and space-filling bearings

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Abstract. We present high precision numerical values for the fractal dimensions of Apollonian packing and various space-filling bearings which might have application for mechanical gearworks, for turbulence or for tectonic motion. We find for the simplest Apollonian packing $d_f = 1.305\,684 \pm 0.000\,010$.

Tiling space with circles by putting them iteratively in each hole between three mutually touching circles and the circle that tangentially touches all three is an old problem often known under the name of 'Apollonian packing' (see figure 1(*a*)). It dates back to Apollonius of Perga who lived around 200 BC and much work has been done since as briefly presented for instance by Mandelbrot [1]. The space left over between circles is a fractal but despite much effort it has not yet been possible to determine the value of the fractal dimension analytically. Only rigorous bounds are known [2, 3] $1.300\,197 < d_f < 1.314\,534$ and Mandelbrot [1] cites a numerical estimate of $d_f \approx 1.3058$ as due to Boyd.

Is it also possible to tile a plane with circles touching one another, called 'space-filling bearings' (SFB), such that all the area is covered with circles (see figure 1(*b*)) [3, 4] of all sizes. This rather exotic question can arise in various contexts. One could imagine the circles to be eddies on the surface of an incompressible fluid and then ask if the fluid motion can be totally decomposed into stable eddies. This could be thought of as a simple model for turbulence [5]. Or, one could think of mechanical roller bearings between two moving surfaces, like two tectonic plates, and then ask if one can completely fill the space between the rolling cylinders with other rolling cylinders such that no cylinder exerts any frictional work on another one. This could explain the behaviour of seismic gaps [6].

We will in the following describe how different SFB packings are constructed and classified for which Apollonian packing comes out as a special case. What follows concentrates on the numerical calculation of porosities and size distributions of the packings. Using the ratio method we get accuracies for the fractal dimension in the fourth digit. Finally we conclude.

We first describe the construction of SFB packings in general following the algorithm introduced in [3, 4]. This is done using circular inversions which is a particular case of Möbius transformations [7]. In an inversion around a circle of radius r a point is mapped into its image point such that both points lie on the same line connecting the centre to infinity and their distances d and d' from the centre fulfil $d \cdot d' = r^2$. Using this relation one gets that a circle of radius r and centre at (x, y) is mapped into

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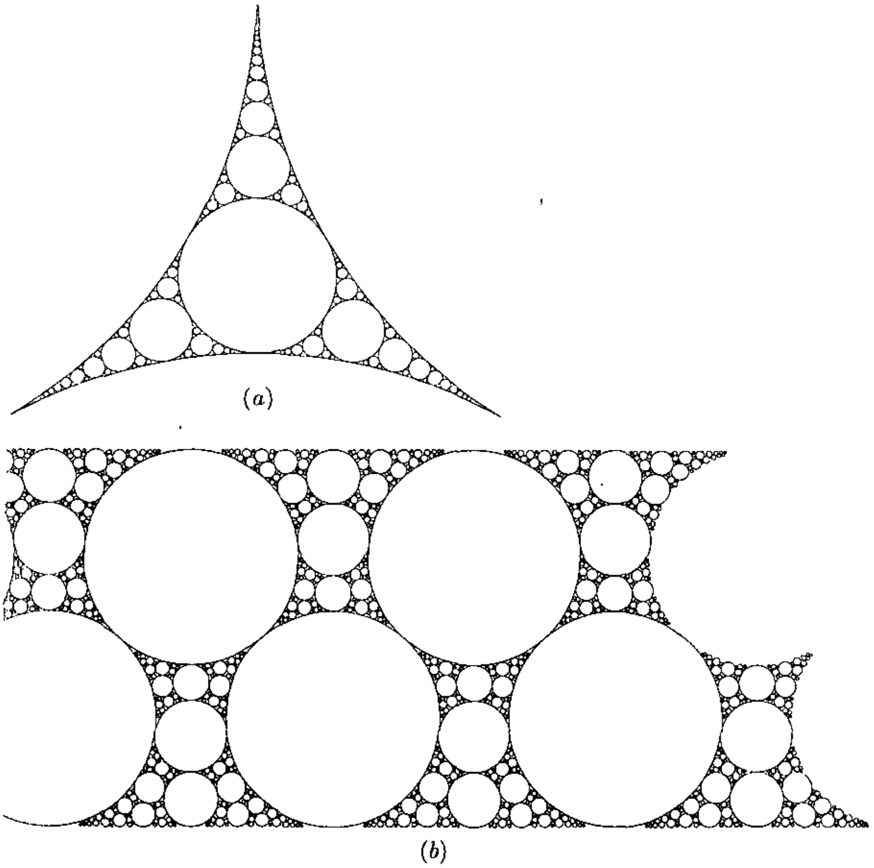

Figure 1. (*a*) Apollonian packing, (*b*) space-filling bearing of the first family for $n = m = 0$.

another circle of radius r' and centre at (x', y') under a circular inversion around a circle of radius r_c and centre at (x_c, y_c) as

$$x' = x_c + \frac{r_c^2}{d^2 - r^2}(x - x_c) \qquad y' = y_c + \frac{r_c^2}{d^2 - r^2}(y - y_c) \tag{1}$$

and

$$r' = r\frac{r_c^2}{d^2 - r^2} \tag{2}$$

where d is the distance between (x, y) and (x_c, y_c).

In a space filled with mutually touching circles one can form a loop which passes through successive touching points of circles. In the case of SFB each circle rolls on the neighbouring touching circles. Therefore for a slipless motion each circle has to move in the opposite direction of the neighbouring circles. This condition imposes the restriction that loops in SFB must have an even number of circles [3, 4]. We will consider here the construction of SFB packings on the strip geometry with only fourfold loops. All these packings are periodic along the strip length.

On the strip geometry one starts with two horizontal lines at unit distance apart and within them places two seed circles A and B touching each other and touching the top and the bottom lines respectively. The parallel lines can be thought of as two circles of infinite radii touching at infinity and therefore along with A and B one gets the initial four-loop configuration with four circles. All four-loop packings on a strip are divided into two families. In the first family the sum of the radii A and B is greater than the strip width and in the second family it is equal to the strip width [3, 4].

In the first family there are an infinite number of ways one can fill a strip. Each member of the family is characterized by two non-negative integer parameters n and m. Given the values of n and m one obtains a set of constants, namely, R_A and R_B the radii of the seed circles A and B, r_A and r_B the radii of the inversion circles and the period $2a$ [3, 4]. Relations of these constants and n and m are discussed later. We first describe the construction of an arbitrary member n and m, assuming these constants are known.

We describe in the following the construction of one unit of packing on the strip whose repetition fills the whole strip. This is done by generating two trees of circles following A and B. As these two trees are independent we construct them one after another and join them together afterwards. We fix our x axis on the bottom line and the y axis passes through the centre of the circle A. From the seed circle A we construct the first generation of circles as in figure 2.

The circle A (with radius R_A) is first inverted around an inversion circle of radius r_A ($\pm R_A$) and centred at $(2a, 1)$ to get the circle A_1 (see figure 2(a)). This circle A_1 is then reflected around the reflection plane at $x = a$ to get the circle A_2. This circle A_2 is then again inverted at the same inversion circle to get A_3. This process of alternate inversion and reflection is continued until one reaches the invariant circle which on further iteration (inversion or reflection) produces the same circle. For a particular value of n one gets $(n+1)$ circles (including the seed A) to reach the invariant circle after n iterations. All circles generated until now belong to the first generation. In figure 2(a) we describe the construction for $n = 3$. The first generation circles are then inverted around a circle of radius r_B centred at $(3a, 0)$. The resulting circles are then reflected around the plane $x = 2a$. These successive inversions and reflections are continued $(m+1)$ times to get the invariant circles. All circles generated until now after first generation constitute the second generation of circles. Next all circles generated so far are alternately inverted and reflected around the inversion circle of radius r_A centred at $(4a, 1)$ and the reflection plane at $x = 3a$ to get the third generation of circles. This process continues for many generations to produce the A tree.

Next we put the seed circle B at (a, R_B) (see figure 2(b)). For the first generation we produce m more circles by successive inversion around an inversion circle of radius r_B centred at $(3a, 0)$ and a reflection plane at $x = 2a$. In the same way as before higher generation circles are produced by shifting the inversion centres and reflection planes by a to the right by alternately changing the inversion circle radii between r_A and r_B and placing thém at top and bottom line to produce the B tree.

Finally we superpose the A and B tree of circles (see figure 2(c)) and then the circles are shifted depending on the generation they were produced by proper translations to place them within $x = 0$ and $x = 3a$ (by mod $2a$) (see figure 2(c)). This constitutes the repetition unit of the packing and the whole packing of the strip is generated by successive translation of this unit to the right and left.

To uniquely define a packing one just needs the two integer parameters n and m (see figure 3). Given them, one gets the radii of the seed circles R_A and R_B, the radii

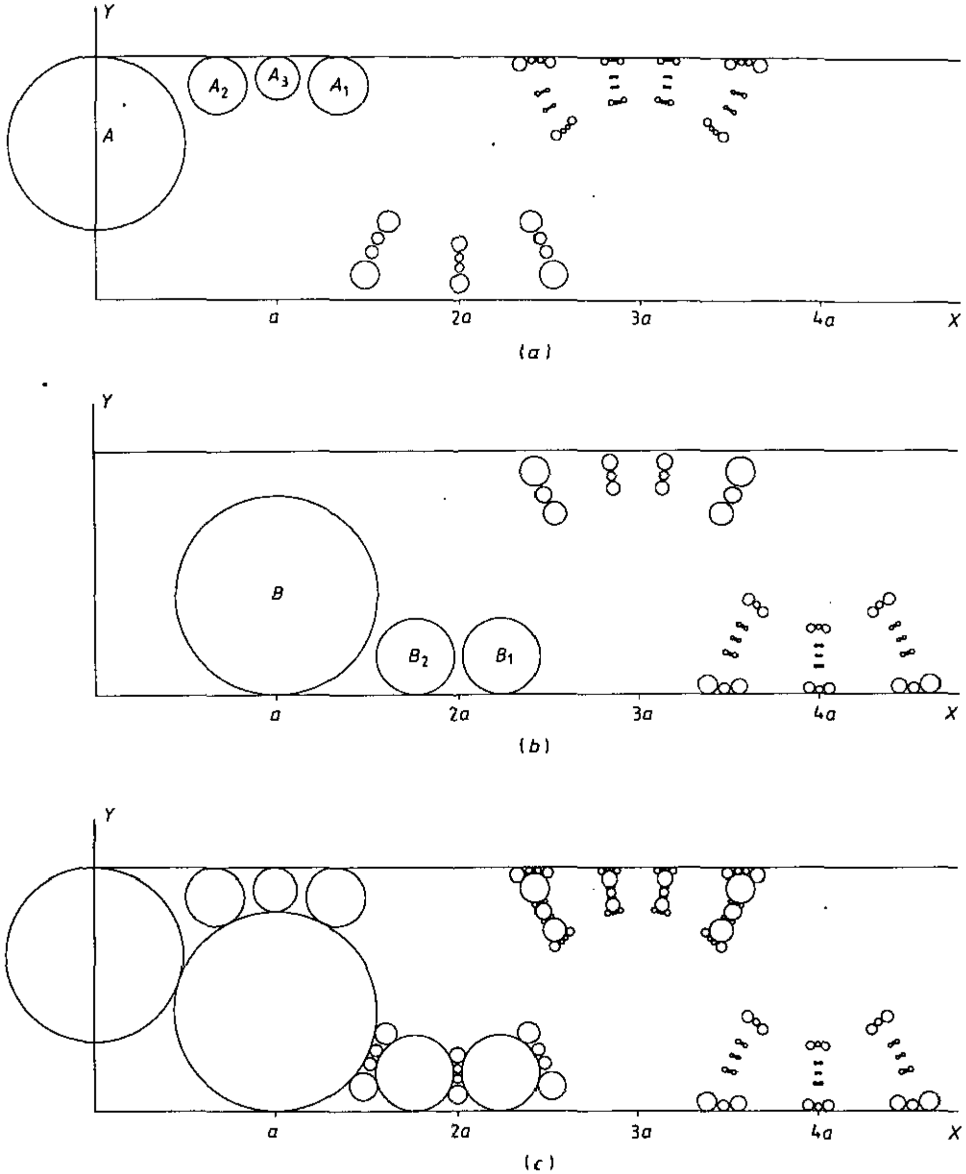

Figure 2. Construction of the $n=3$, $m=2$ SFB packing. (a) and (b) show circles produced up to several generations starting from the seed circles A and B. In (c) these two trees are superposed.

of the inversion circles r_A and r_B and the period $2a$, fixed numbers. Calculation of these constants uses the fact that for the first generation after nth iteration (successive inversion and reflection) one gets the invariant circle of A and after mth iteration the invariant circle of B is obtained (see [3, 4] for derivation of these equations).

Let us define $z_n = \cos^{-2}[\pi/(n+3)]$. One then obtains for the half period a of the first family

$$a^{-2} = z_n + z_m - 1 \tag{3}$$

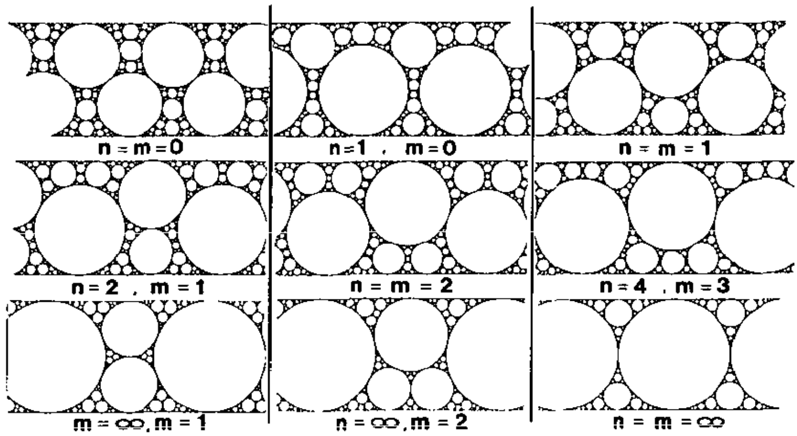

Figure 3. Nine different combinations of n and m of the first family.

and the radii of inversions

$$r_{A,B}^2 = \frac{z_{n,m}}{z_n + z_m - 1} \tag{4}$$

and the radii of the circles A and B are given by

$$2R_A = r_A^2 \qquad \text{and} \qquad 2R_B = r_B^2. \tag{5}$$

It is interesting to note that for $n = m = \infty$ and for $n = 0$, $m = \infty$ one obtains the classical Apollonian packing. In fact the whole one-parameter family with $n = \infty$ and arbitrary m has threefold loops and is in this sense the full 'Apollonian family'. It is obtained in this case as the limit in which one of the four circles in each loop becomes infinitely small.

The packings just constructed are evidently fractal. One way to define their fractal dimensions is by introducing a cut-off length ε such that one considers in a packing exactly those circles that have a radius larger than ε. Now one can calculate (on the computer) the number $N(\varepsilon)$ of circles per unit area, the sum $s(\varepsilon)$ of the perimeters of the circles ('surface') per unit area and the 'porosity' $p(\varepsilon)$, i.e. the area that is not covered by circles per unit area. All these quantities can be related to the distribution $n(r)$ of radii r, i.e. the number of circles of radius r per unit area through:

$$N(\varepsilon) = \int_\varepsilon^\infty n(r)\,\mathrm{d}r \tag{6a}$$

$$s(\varepsilon) = 2\pi \int_\varepsilon^\infty rn(r)\,\mathrm{d}r \tag{6b}$$

$$p(\varepsilon) = 1 - \pi \int_\varepsilon^\infty r^2 n(r)\,\mathrm{d}r. \tag{6c}$$

If $n(r)$ can be described by a simple power law

$$n(r) \sim r^{-\bar{\tau}} \tag{7}$$

then one finds

$$N(\varepsilon) \sim \varepsilon^{-d_f}$$
$$s(\varepsilon) \sim \varepsilon^{1-d_f} \qquad \text{with } d_f = \tilde{\tau} - 1 \tag{8}$$
$$p(\varepsilon) \sim \varepsilon^{2-d_f}$$

where d_f is the fractal dimension [3].

We have calculated the distribution of radii of various packings of the first family up to $\varepsilon = 2^{-22}$ by producing around 0.5 to 1.5×10^8 circles using a fully vectorized, memory-saving algorithm following the lines used previously. We spent for each packing about nine hours on one Cray-YMP processor at HLRZ.

In figure 4(a) we show N, s and p plotted double logarithmically against the cut-off for $n = m = 0$. The straight lines over several orders of magnitude confirm the power-law

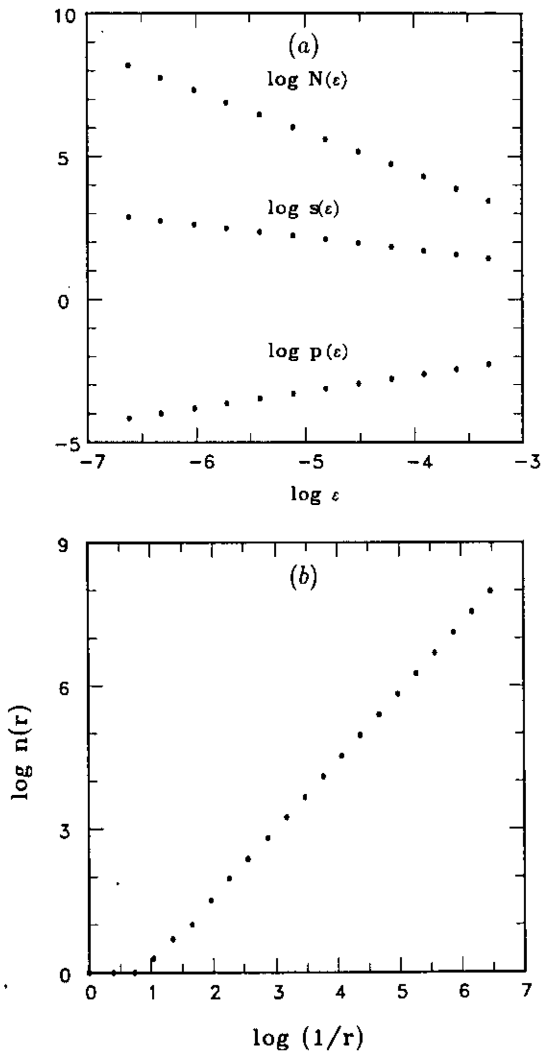

Figure 4. Log–log plots of: (a) the number of circles N, the surface s and the porosity p as function of the cut-off ε; (b) disk-radius distribution $n(r)$ as function of the inverse radius r. Both figures are for $n = m = 0$ of the first family.

behaviour. The fact that the porosity goes to zero with ε is a numerical verification that the packings are space-filling.

In figure 4(b) we see the disk radius distribution $n(r)$ itself, binned in powers of two, plotted double logarithmically against the inverse of the radius r. For large radii the events are more rare and the binning produces deviations from the straight line that we might call corrections to scaling. All these calculations are done in the quartic precision (real words of 16 bytes) so as to avoid any contamination of numerical inaccuracy. Therefore our data are as good as getting them exactly. We give the data for the distribution $n(r)$ in table 1 for the members $(0, 0)$ and (∞, ∞).

We analysed our data by the methods used in the series analysis of exact enumeration data. Though the double logarithmic plots of the three moments and the distribution function $n(r)$ are nice straight lines, we calculated the successive slopes of these curves which using (8) give effective fractal dimensions d_{eff}. In figure 5 we see d_{eff} is plotted against $1/l$ where d_{eff} is obtained from successive slopes of $N(\varepsilon)$, $s(\varepsilon)$ and $p(\varepsilon)$ for the values of 2^{-l+1} and 2^{-l} and for $n(r)$ we take the successive slopes of the lth and the $(l-1)$th bin, for $n = m = 0$ and the Apollonian case $n = m = \infty$. We see that the values of d_{eff} obtained from $N(\varepsilon)$, $p(\varepsilon)$ and $n(r)$ fluctuate widely for small l values but this fluctuation decreases rapidly with increasing l and approaches the same asymptotic d_{f} value for different curves. However d_{eff} values obtained from $s(\varepsilon)$ monotonically decrease with increasing l and approach the value obtained from other moments but even at the largest value of l it gives a much higher value and therefore we discard this result in estimating d_{f}. To calculate the asymptotic value of d_{f} we average over the d_{eff} values for the largest l $(=22)$ for $N(\varepsilon)$, $p(\varepsilon)$ and $n(r)$ and for

Table 1. The distribution of circle radii $n(r)$ as a function of the inverse radius $(1/r)$. Second and third columns give the number of circles whose inverse radii are within 2^{l-1} and $(2^l - 1)$ for $n = m = 0$ and $n = m = \infty$.

l	$(0, 0)$	(∞, ∞)
1	0	0
2	1	1
3	0	2
4	2	0
5	5	10
6	10	14
7	32	44
8	92	88
9	238	250
10	648	570
11	1 748	1 470
12	4 666	3 606
13	12 702	8 938
14	34 092	21 830
15	92 102	54 210
16	248 278	134 018
17	670 614	331 778
18	1 809 056	818 054
19	4 882 558	2 024 740
20	13 175 816	5 003 640
21	35 546 746	12 367 412
22	95 924 460	30 575 024

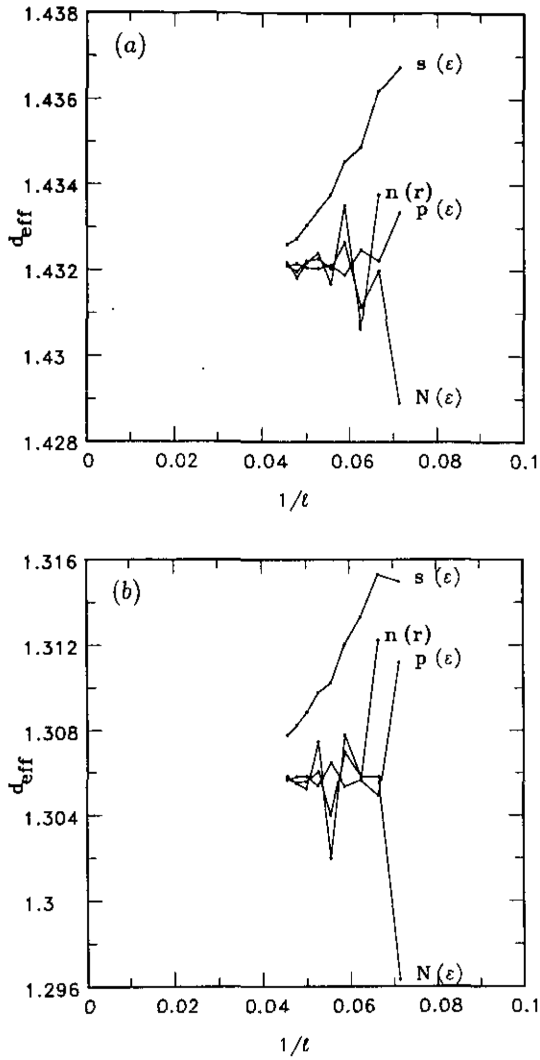

Figure 5. Effective fractal dimension d_{eff} as a function of the binning index l for (*a*) $n = m = 0$ and (*b*) $n = m = \infty$.

Table 2. Fractal dimensions obtained for various packings of the first family.

n	m	d_f
0	0	1.4321 ± 0.0005
0	1	1.4057 ± 0.0005
1	1	1.4123 ± 0.0005
∞	2	1.3397 ± 0.0005
∞	∞	1.3057 ± 0.0005

the error we put the half of the difference in maximum and minimum of d_{eff} for the last three or four l values. In table 2 we present our final values for various packings.

The fractal dimension one obtains for the Apollonian packing, i.e. $n=m=\infty$, agrees well with Boyd's value [1]. For the first family the fractal dimension obtained continuously increases with decreasing n and m. It is interesting to note that the fractal dimension is not constant within the Apollonian family (n, ∞).

If instead of looking at all circles larger than ε one takes all circles produced up to a given level of the iteration then multifractality of the Apollonian packing has been observed [9]. Also the set of the touching points has been studied and it was found [10] to be multifractal for a fixed number of iterations but not multifractal for fixed cut-off.

In [3] the observation was made that numerically the fractal dimensions one obtains from the various moments of (5) did slowly decrease with increasing order of the moment. We think that this was due to numerical inaccuracies in taking into account eventual curvatures coming from corrections to scaling.

We have seen that the packings are fractal with dimensions that for the first family lie between 1.3057 and 1.4321. The distribution of radii is numerically compatible with a power law $n(r) \sim r^{-\bar{\tau}}$. Since all discs rotate with the same tangential velocity v we can calculate the kinetic energy $E(r)$ of discs of radius r as $E(r) = \pi \rho v^2 r^2 n(r)$ (ρ is the density of the material) and from there the energy spectrum $E(k)\,\mathrm{d}k$ as a function of the wavevector $k = r^{-1}$. We find:

$$E(k)\,\mathrm{d}k \sim k^{\bar{\tau}-4}\,\mathrm{d}k = k^{d_f-3}\,\mathrm{d}k. \tag{9}$$

Using the values of d_f calculated here we see that although none is exactly 4/5 their values are not far from Kolmogoroff scaling of the energy spectrum of homogeneous fully developed turbulence [6]: $E(k)\,\mathrm{d}k \sim k^{-5/3}\,\mathrm{d}k$. This might be a coincidence but might also have a more profound meaning, giving a geometrical interpretation of turbulence in the picture of the transfer of energy to smaller and smaller eddies in the inertial regime. One could also imagine a random mixture of all the packings [4] such that the effective dimension actually coincides with 4/3.

We would like to thank P Grassberger for sending us the computer program for Melzak's algorithm and for many useful discussions. We also thank the referee for useful comments.

Note added in proof. After submission of this paper, P Grassberger pointed out a very simple algorithm by Melzak [11] for generating packing patterns for the Apollonian case $n=m=\infty$. Using this algorithm we simulated Apollonian packings up to $\varepsilon = 2^{-25,-26,-27}$, the last one using 5.23 days of Sun4 CPU. From these new data we conclude a value of 1.305 684 ± 0.000 010 for the fractal dimension of the Apollonian packing.

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